AN EXISTENCE THEOREM FOR PERIODIC SOLUTIONS*

BY

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Introduction.

Under certain circumstances differential equations which are analytic functions of a parameter and of the dependent variables admit of periodic solutions which are expansible as converging power series in the parameter. The proof of the existence of the periodic solutions is usually much more laborious than the actual construction of their first terms. Heretofore it has been necessary to give the proof of their existence in each specific case before the validity of the construction was assured.

In this paper the problem of proving the existence of the periodic solutions in a very general class of cases will be developed in such a manner that it is closely related to that of their practical construction; and it will be shown from these relations that, whatever the multiplicity of the solutions, when the formal construction is possible the existence proof can be made. Therefore in this class of cases the direct existence proof is no longer necessary, and the complete discussion is correspondingly simplified and shortened.

The Differential Equations.

Let us consider the differential equations

$$\frac{dy_i}{dt} = f_i(y_1, \cdots, y_n, \mu; t) \qquad (i = 1, \cdots, n), \tag{1}$$

where the f_i are uniform, continuous, and periodic functions of t. Without loss of generality the period can be taken equal to 2π. Now suppose that when the parameter μ is zero, equations (1) admit the periodic solution

$$y_i = y_i^{(0)}(t) \qquad (i = 1, \cdots, n),$$

in which the period is 2π, or a multiple of 2π. If the period is a multiple of 2π, we may change t by such a constant factor that it reduces to 2π.

In order to discuss the solutions of (1) when μ is distinct from zero, let

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$$y_i = y_i^{(0)} + x_i \qquad (i = 1, \cdots, n).$$

Suppose the f_i are analytic functions of $y_i - y_i^{(0)} = x_i$ and μ, and regular at

$$y_i - y_i^{(0)} = \mu = 0$$

for $0 \gtreqless t \leqq 2\pi$. We notice that the right members of the differential equations resulting from this transformation vanish for

$$x_i = \mu = 0.$$

If for convenience of notation we denote the parameter μ by x_0, the right members of equations (1) can be expanded as power series in the x_j with coefficients which are periodic in t, as follows:

$$\frac{dx_i}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_j = \theta_0^{(i)} x_0 + \sum_{j=0}^{n} \sum_{k=0}^{j} \theta_{jk}^{(i)} x_j x_k + \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{l=0}^{k} \theta_{jkl}^{(i)} x_j x_k x_l + \cdots \qquad (i = 1, \cdots, n), \tag{2}$$

where the $\theta_{j\cdots}^{(i)}$ are continuous periodic functions of t.

Before integrating equations (2) let us consider the equations

$$\frac{dx_i}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_j = 0, \tag{3}$$

which are obtained by setting the left members of (2) equal to zero. These equations are linear and homogeneous with periodic coefficients. A single equation of the nth order with periodic coefficients was treated by FLOQUET,* and the set (3) was discussed in detail by MOULTON and MACMILLAN.† The solutions of these equations can be written in the form

$$x_i = \sum_{j=1}^{n} A_j \xi_j^{(i)}(t) \qquad (i = 1, \cdots, n), \tag{4}$$

in which the A_j are constants of integration, while the $\xi_j^{(i)}$ can be so taken that the determinant

$$\Delta = | \xi_j^{(i)}(0) |$$

equals unity. The determinant Δ expanded according to the elements of the ith line is

$$\Delta = \sum_{j=1}^{n} \Delta_j^{(i)} \xi_j^{(i)}(0), \tag{5}$$

where $\Delta_j^{(i)}$ is the minor obtained by suppressing the ith line and jth column of Δ and attaching the proper sign.

*Annales de l'École Normale Supérieure, vol. 12 (1883–1884), p. 47.

†American Journal of Mathematics, vol. 33 (1911), pp. 63–96.

Integration of the Differential Equations as Power Series in the Initial Values.

If the initial values of the x_i be denoted by α_i, and if x_0 be denoted by α_0, it follows from Cauchy's existence theorem* that the solutions of (2) are expansible as power series in the α_j $(j = 0, \cdots, n)$, reducing for $t = 0$ to

$$x_i(0) = \alpha_i;$$

and it follows from Poincaré's extension † of Cauchy's theorem that these solutions converge for all values of t in any preassigned range for which the expansions (2) converge, provided the moduli of the α_j are sufficiently small. That is, the solutions of (2) can be written

$$x_i = \sum_{j=0}^{n} x_j^{(i)} \alpha_j + \text{higher degree terms} \qquad (i = 1, \cdots, n), \tag{6}$$

where

$$x_i^{(i)}(0) = 1 \qquad (i = 1, \cdots, n),$$

and all the remaining coefficients vanish at $t = 0$.

On substituting (6) in (2) and rearranging as power series in $\alpha_0, \cdots, \alpha_n$, we have from the coefficients of α_j

$$\frac{dx_j^{(i)}}{dt} + \sum_{k=1}^{n} \theta_k^{(i)} x_j^{(k)} = 0 \qquad (i, j = 1, \cdots, n). \tag{7}$$

These equations are the same as (3), and the solutions are therefore

$$x_j^{(i)} = \sum_{k=1}^{n} A_k^{(j)} \xi_k^{(i)}(t) \qquad (i, j = 1, \cdots, n). \tag{8}$$

From the initial conditions it follows that

$$x_j^{(i)}(0) = \delta_{ij},$$

where

$$\delta_{ij} = 0 \quad (i \neq j), \qquad \delta_{jj} = 1 \quad (i = j).$$

Hence

$$\sum_{k=1}^{n} A_k^{(j)} \xi_k^{(i)}(0) = \delta_{ij} \qquad (i = 1, \cdots, n).$$

Since the determinant of the left members, Δ, is unity, the solutions of these equations are

$$A_k^{(j)} = \Delta_k^{(j)},$$

so that

$$x_j^{(i)} = \sum_{k=1}^{n} \Delta_k^{(j)} \xi_k^{(i)}(t) \qquad (i, j = 1, \cdots, n). \tag{9}$$

* *Collected Works*, 1st series, vol. 7.

† *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, p. 55.

Since $x_0 = \alpha_0$, the coefficients of α_0 are linear but not homogeneous. From equations (2) we find

$$\frac{dx_0^{(i)}}{dt} + \sum_{k=1}^{n} \theta_k^{(i)} x_0^{(k)} = \theta_0^{(i)}(t) \qquad (i = 1, \cdots, n). \tag{10}$$

The general solution of these non-homogeneous equations can be found by the method of variation of parameters. We can denote it by

$$x_0^{(i)} = m_i(t) + \sum_{k=1}^{n} A_k^{(0)} \xi_k^{(i)} \qquad (i = 1, \cdots, n), \tag{11}$$

where the $m_i(t)$ are particular solutions, and the $A_k^{(0)}$ are constants of integration.

From the initial conditions we must have

$$x_0^{(i)}(0) = 0.$$

Hence

$$\sum_{k=1}^{n} A_k^{(0)} \xi_k^{(i)}(0) = -m_i(0) \qquad (i = 1, \cdots, n). \tag{12}$$

Solving these equations, we get

$$A_k^{(0)} = -\sum_{j=1}^{n} \Delta_k^{(j)} m_j(0) \qquad (k = 1, \cdots, n),$$

so that

$$x_0^{(i)} = m_i(t) - \sum_{k=1}^{n} \sum_{j=1}^{n} \Delta_k^{(j)} m_j(0) \xi_k^{(i)}(t) \qquad (i = 1, \cdots, n). \tag{13}$$

The sum of all of the linear terms in the solution for the x_i is found from (9) and (13) to be

$$\begin{aligned} \sum_{j=0}^{n} x_j^{(i)} \alpha_j &= \sum_{j=1}^{n} \sum_{k=1}^{n} \Delta_k^{(j)} \xi_k^{(i)} \alpha_j - \sum_{k=1}^{n} \sum_{j=1}^{n} \Delta_k^{(j)} m_j(0) \xi_k^{(i)} \alpha_0 + m_i(t) \cdot \alpha_0 \\ &= \sum_{k=1}^{n} \left[\sum_{j=1}^{n} \Delta_k^{(j)} [\alpha_j - m_i(0) \alpha_0] \right] \xi_k^{(i)}(t) + m_i(t) \cdot \alpha_0. \end{aligned} \tag{14}$$

A Change of Parameters.

Let us now make a change of parameters by the linear substitution

$$\alpha_0 = \beta_0, \qquad \sum_{j=1}^{n} \Delta_k^{(j)} [a_j - m_i(0) \alpha_0] = \beta_k \qquad (k = 1, \cdots, n), \tag{15}$$

which can be solved for the α's in terms of the β's, since the determinant $| \Delta_k^{(j)} |$ has the value unity. The result is

$$\alpha_0 = \beta_0, \qquad \alpha_j = m_j(0) \beta_0 + \sum_{k=1}^{n} \xi_k^{(j)}(0) \beta_k \qquad (j = 1, \cdots, n). \tag{16}$$

Since the solutions (6) as power series in $\alpha_0, \cdots, \alpha_n$ converge provided the moduli of the α_j are sufficiently small, they can also be developed as power series in the β_j which will converge provided the moduli of the β_i are sufficiently small; and for $t = 0$ will reduce to

$$x_i(0) = m_i(0)\beta_0 + \sum_{k=1}^{n} \xi_k^{(i)}(0)\beta_k \qquad (i = 1, \cdots, n).$$

Since the transformation (15) is linear and homogeneous, the coefficient of every β of degree higher than the first will vanish at $t = 0$. Expressed in terms of the β's, the solutions can be written

$$\begin{aligned} x_i = m_i(t)\beta_0 + \sum_{j=1}^{n} \xi_j^{(i)}(t)\beta_j + \sum_{j=0}^{n}\sum_{k=0}^{j} x_{jk}^{(i)}\beta_j\beta_k \\ + \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k} x_{jkl}^{(i)}\beta_j\beta_k\beta_l + \cdots \qquad (i = 1, \cdots, n). \end{aligned} \tag{17}$$

Since $\beta_1, \cdots, \beta_n$ can be regarded as arbitrary, one sees that they are nothing else than the constants of integration of the linear terms (3). We have therefore, since the periodic properties of (1) have not been used, the

THEOREM: *If*

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n; \mu; t) \qquad (i = 1, \cdots, n) \tag{A}$$

is a system of differential equations in which the f_i are analytic functions of $x_1, \cdots, x_n$; μ, regular at the point $x_1 = \cdots = x_n = \mu = 0$, for all t in the interval $0 \gtreqless t \gtreqless T$, and vanish for $x_1 = \cdots = x_n = \mu = 0$, and if the f_i are uniform and continuous in t in the interval $0 \gtreqless t \gtreqless T$; and if

$$\frac{dx_i}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_j = 0 \qquad (i = 1, \cdots, n) \tag{B}$$

are the linear terms of (A) equated to zero, then the solutions of (A) are expansible as power series in μ and the constants of integration of the solutions of (B), and these solutions converge in the interval $0 \gtreqless t \gtreqless T$ provided the moduli of the parameter μ and the constants of integration are sufficiently small.

For the sake of future notation we will write (17) in the form

$$x_i = \sum_{j=0}^{n} x_j^{(i)}\beta_j + \sum_{j=0}^{n}\sum_{k=0}^{j} x_{jk}^{(i)}\beta_j\beta_k + \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k} x_{jkl}^{(i)}\beta_j\beta_k\beta_l + \cdots, \tag{18}$$

though it is to be observed that the $x_j^{(i)}$ are the same as the linear terms of (17), viz:

$$x_j^{(i)} = \xi_j^{(i)} \quad (i, j = 1, \cdots, n), \qquad x_0^{(i)} = m_i \quad (i = 1, \cdots, n),$$

and are not the same as the $x_j^{(i)}$ used in (6)–(14).

On substituting (18) in (2) and rearranging as power series in $\beta_0, \cdots, \beta_n$, we find from the coefficients of the terms of the second degree

$$
\begin{aligned}
\sum_{j=0}^{n}\sum_{k=0}^{j}\left[\frac{dx_{jk}^{(i)}}{dt}+\sum_{l=1}^{n}\theta_l^{(i)}x_{jk}^{(l)}\right]\beta_j\beta_k &= \sum_{j=0}^{n}\sum_{k=0}^{j}\theta_{jk}^{(i)}\left(\sum_{r=0}^{n}x_r^{(j)}\beta_r\right)\left(\sum_{s=0}^{n}x_s^{(k)}\beta_s\right)\\
(19)\qquad &= \sum_{r=0}^{n}\sum_{s=0}^{r}\epsilon_{rs}\left(\sum_{j=0}^{n}\sum_{k=0}^{j}\theta_{jk}^{(i)}(x_r^{(j)}x_s^{(k)}+x_r^{(k)}x_s^{(j)})\right)\beta_r\beta_s\\
&= \sum_{j=0}^{n}\sum_{k=0}^{j}\epsilon_{jk}\left(\sum_{r=0}^{n}\sum_{s=0}^{r}\theta_{rs}^{(i)}(x_j^{(r)}x_k^{(s)}+x_j^{(s)}x_k^{(r)})\right)\beta_j\beta_k,
\end{aligned}
$$

where $\epsilon_{jk} = 1$ if $j \neq k$, and $\epsilon_{jj} = 1/2$. The terms

$$\sum_{r=0}^{n}\sum_{s=0}^{r}\theta_{rs}^{(i)}(x_j^{(r)}x_k^{(s)}+x_j^{(s)}x_k^{(r)})$$

are all known as soon as the linear terms are known. Their sum depends only on the indices i, j, k. Consequently the particular solution of (19), which depends on this sum, can be denoted by $\varphi_{jk}^{(i)}(t)$, and the solution of (19) is

$$(20)\qquad \sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(i)}\beta_j\beta_k = \sum_{j=0}^{n}\sum_{k=0}^{j}\left[\varphi_{jk}^{(i)}(t)+\sum_{l=1}^{n}A_{jk}^{(l)}\xi_l^{(i)}\right]\beta_j\beta_k.$$

By virtue of the initial conditions we must have

$$\sum_{l=1}^{n}A_{jk}^{(l)}\xi_l^{(i)}(0) = -\varphi_{jk}^{(i)}(0)\qquad (i = 1, \cdots, n),$$

whence

$$A_{jk}^{(l)} = -\sum_{h=1}^{n}\Delta_l^{(h)}\varphi_{jk}^{(h)}(0),$$

so that (20) becomes

$$(21)\quad \sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(i)}\beta_j\beta_k = \sum_{j=0}^{n}\sum_{k=0}^{j}\left[\varphi_{jk}^{(i)}(t)-\sum_{l=1}^{n}\sum_{h=1}^{n}\Delta_l^{(h)}\varphi_{jk}^{(h)}(0)\xi_l^{(i)}(t)\right]\beta_j\beta_k.$$

In the same manner we obtain for the terms of the third degree

$$
\begin{aligned}
&\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\frac{dx_{jkl}^{(i)}}{dt}+\sum_{m=1}^{n}\theta_m^{(i)}x_{jkl}^{(m)}\right]\beta_j\beta_k\beta_l\\
&= \sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q}\theta_{pqr}^{(i)}\left(\sum_{j=0}^{n}x_j^{(p)}\beta_j\right)\left(\sum_{k=0}^{n}x_k^{(q)}\beta_k\right)\left(\sum_{l=0}^{n}x_l^{(r)}\beta_l\right)+\sum_{p=0}^{n}\sum_{q=0}^{p}\theta_{pq}^{(i)}\\
&\quad\times\left\{\left(\sum_{j=0}^{n}x_j^{(q)}\beta_j\right)\left(\sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(p)}\beta_j\beta_k\right)+\left(\sum_{j=0}^{n}x_j^{(p)}\beta_j\right)\left(\sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(q)}\beta_j\beta_k\right)\right\}
\end{aligned}
$$

$$(22)\quad = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q}\epsilon_{jkl}\,\theta_{pqr}^{(i)}\left\{\begin{array}{l} +\,(x_j^{(p)}x_k^{(q)}+x_k^{(p)}x_j^{(q)})\,x_l^{(r)} \\ +\,(x_j^{(p)}x_l^{(q)}+x_l^{(p)}x_j^{(q)})\,x_l^{(r)} \\ +\,(x_k^{(p)}x_l^{(q)}+x_l^{(p)}x_k^{(q)})\,x_j^{(r)} \end{array}\right\} + \sum_{p=0}^{n}\sum_{q=0}^{p}\delta_{jkl}\,\theta_{pq}^{(i)}\left\{\begin{array}{l} +\,(x_{jk}^{(p)}x_l^{(q)}+x_{jk}^{(q)}x_l^{(p)}) \\ +\,(x_{jl}^{(p)}x_k^{(q)}+x_{jl}^{(q)}x_k^{(p)}) \\ +\,(x_{kl}^{(p)}x_j^{(q)}+x_{kl}^{(q)}x_j^{(p)}) \end{array}\right\}\right]\beta_j\beta_k\beta_l .$$

The symbol ϵ_{jkl} means 1 if j, k and l are all different; 1/2 if two are alike and different from the third; and 1/6 if all three are alike. The symbol δ_{jkl} means that when two of the three letters j, k, and l are equal only one of two like terms is to be taken, and when all three are equal only one of three like terms is to be taken.

The right members of equations (22) are all known and they depend only upon the indices i, j, k, and l. If therefore we denote the particular solutions by $\varphi_{jkl}^{(i)}(t)$, the complete solutions can be written

$$(23)\quad \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k} x_{jkl}^{(i)}\beta_j\beta_k\beta_l = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\varphi_{jkl}^{(i)}(t)+\sum_{m=1}^{n}A_{jkl}^{(m)}\xi_m^{(i)}\right]\beta_j\beta_k\beta_l .$$

On determining the constants of integration $A_{jkl}^{(m)}$ so as to make the $x_{jkl}^{(i)}$ vanish at $t = 0$, we get

$$(24)\quad \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k} x_{jkl}^{(i)}\beta_j\beta_k\beta_l = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\varphi_{jkl}^{(i)}(t)-\sum_{g=1}^{n}\sum_{h=1}^{n}\Delta_g^{(h)}\varphi_{jkl}^{(h)}(0)\,\xi_g^{(i)}\right]\beta_j\beta_k\beta_l \qquad (i = 1, \cdots, n).$$

The coefficients of the fourth and higher degree terms in $\beta_1, \cdots, \beta_n$ can be developed in a similar manner.

Integration of the Differential Equations as Power Series in μ.

Without specifying the initial values of the x_i and without regard to the convergence of the series so derived, the differential equations can be integrated formally as power series in μ, or any root of μ, say $\mu^{1/p}$. We will integrate as power series in μ, and leave the constants of integration arising at each step undetermined. We shall have then

$$(25)\quad x_i = x_1^{(i)}\mu + x_2^{(i)}\mu^2 + x_3^{(i)}\mu^3 + \cdots \qquad (i = 1, \cdots, n).$$

Substituting these expressions in (2), we have from the coefficients of the first power of μ, since $x_0 = \mu$,

$$(26)\quad \frac{dx_1^{(i)}}{dt} + \sum_{j=1}^{n}\theta_j^{(i)}x_1^{(j)} = \theta_0^{(i)}(t) \qquad (i = 1, \cdots, n).$$

These equations are the same as equations (10). They have therefore the same solution (11), viz.,

$$x_1^{(i)} = m_i(t) + \sum_{j=1}^{n} A_j^{(i)} \xi_j^{(i)} \qquad (i = 1, \cdots, n), \tag{27}$$

which can be written

$$x_1^{(i)} = \sum_{j=0}^{n} A_j^{(1)} x_j^{(i)} \qquad (i = 1, \cdots, n), \tag{28}$$

the $x_j^{(i)}$ being the same functions of t as in (18), and $A_0^{(i)} = 1$.

The coefficients of μ^2 give the equations

$$\frac{dx_2^{(i)}}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_2^{(j)} = \sum_{j=0}^{n} \sum_{k=0}^{j} \theta_{jk}^{(i)} \left(\sum_{p=0}^{n} A_p^{(1)} x_p^{(j)} \right) \left(\sum_{q=0}^{n} A_q^{(1)} x_q^{(k)} \right) \qquad (i = 1, \cdots, n). \tag{29}$$

These equations are essentially the same as (19), the right members differing only in that $A_p^{(1)}$ is substituted for β_p. The solutions are therefore

$$x_2^{(i)} = \sum_{j=0}^{n} \sum_{k=0}^{j} \varphi_{jk}^{(i)}(t) A_j^{(1)} A_k^{(1)} + \sum_{l=1}^{n} B_l^{(2)} \xi_l^{(i)} \qquad (i = 1, \cdots, n). \tag{30}$$

We propose to leave the constants of integration undetermined, but we observe that if we give the $B_l^{(2)}$ the form

$$B_l^{(2)} = - \sum_{k=1}^{n} \Delta_l^{(h)} \sum_{j=0}^{n} \sum_{k=0}^{j} \varphi_{jk}^{(h)}(0)\, A_j^{(1)} A_k^{(1)} + A_l^{(2)},$$

where the $A_l^{(2)}$ are undetermined, the solution (30) takes the form

$$\begin{aligned} x_2^{(i)} &= \sum_{j=0}^{n} \sum_{k=0}^{j} \left[\varphi_{jk}^{(i)}(t) - \sum_{l=1}^{n} \sum_{h=1}^{n} \Delta_l^{(h)} \varphi_{jk}^{(h)}(0)\, \xi_l^{(i)}(t) \right] A_j^{(1)} A_k^{(1)} + \sum_{j=1}^{n} A_j^{(2)} \xi_j^{(i)} \\ &= \sum_{j=0}^{n} \sum_{k=0}^{j} x_{jk}^{(i)} A_j^{(1)} A_k^{(1)} + \sum_{j=0}^{n} A_j^{(2)} x_j^{(i)}, \end{aligned} \tag{31}$$

where $A_0^{(2)} = 0$. The initial value of $x_2^{(i)}$ is then

$$\sum_{j=0}^{n} A_j^{(2)} x_j^{(i)}(0),$$

which is undetermined since all of the $A_j^{(2)}$ are undetermined.

From the coefficients of μ^3 we have

$$\begin{aligned} \frac{dx_3^{(i)}}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_3^{(j)} &= \sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{r=0}^{q} \theta_{pqr}^{(i)} \left(\sum_{j=0}^{n} A_j^{(1)} x_j^{(p)} \right) \left(\sum_{k=0}^{n} A_k^{(1)} x_k^{(q)} \right) \left(\sum_{l=0}^{n} A_l^{(1)} x_l^{(r)} \right) \\ &+ \sum_{p=0}^{n} \sum_{q=0}^{p} \theta_{pq}^{(i)} \left\{ \sum_{j=0}^{n} A_j^{(1)} x_j^{(p)} \left[\sum_{k=0}^{n} \sum_{l=0}^{k} A_k^{(1)} A_l^{(1)} x_{kl}^{(q)} + \sum_{k=0}^{n} A_k^{(2)} x_k^{(q)} \right] \right. \\ &\left. + \sum_{j=0}^{n} A_j^{(1)} x_j^{(q)} \left[\sum_{k=0}^{n} \sum_{l=0}^{k} A_k^{(1)} A_l^{(1)} x_{kl}^{(p)} + \sum_{k=0}^{n} A_k^{(2)} x_k^{(p)} \right] \right\}, \end{aligned} \tag{32}$$

which becomes on rearranging the right member

$$
\begin{aligned}
\frac{dx_3^{(i)}}{dt}+\sum_{j=1}^{n}\theta_j^{(i)}x_3^{(j)} = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\Bigg[&\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q}\epsilon_{jkl}\theta_{qpr}^{(i)}\left\{\begin{array}{l} +\,(x_j^{(p)}x_k^{(q)}+x_j^{(q)}x_k^{(p)})x_l^{(r)} \\ +\,(x_j^{(p)}x_l^{(q)}+x_j^{(q)}x_l^{(p)})x_k^{(r)} \\ +\,(x_k^{(p)}x_l^{(q)}+x_k^{(q)}x_l^{(p)})x_j^{(r)} \end{array}\right\} \\
(33)\qquad &+\sum_{p=0}^{n}\sum_{q=0}^{p}\delta_{jkl}\theta_{pq}^{(i)}\left\{\begin{array}{l} +\,(x_{jk}^{(p)}x_l^{(q)}+x_{jk}^{(q)}x_l^{(p)}) \\ +\,(x_{jl}^{(p)}x_k^{(q)}+x_{jl}^{(q)}x_k^{(p)}) \\ +\,(x_{kl}^{(p)}x_j^{(q)}+x_{kl}^{(q)}x_j^{(p)}) \end{array}\right\}\Bigg]A_j^{(1)}A_k^{(1)}A_l^{(1)} \\
&+\sum_{j=0}^{n}\sum_{k=0}^{j}\left[\sum_{p=0}^{n}\sum_{q=0}^{p}\epsilon_{jk}\theta_{pq}^{(i)}(x_j^{(p)}x_k^{(q)}+x_j^{(q)}x_k^{(p)})\right][A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)}].
\end{aligned}
$$

On comparing these equations with (22) and (19), it is seen that the solution is

$$
(34)\quad x_3^{(i)} = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\varphi_{jkl}^{(i)}A_j^{(1)}A_k^{(1)}A_l^{(1)}+\sum_{j=0}^{n}\sum_{k=0}^{j}\varphi_{jk}^{(i)}[A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)}]+\sum_{g=1}^{n}B_g^{(3)}\xi_g^{(i)}.
$$

If now the constants of integration are given the form

$$
\begin{aligned}
B_g^{(3)} = -\sum_{h=1}^{n}\Delta_g^{(h)}\Bigg[&\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\varphi_{jkl}^{(h)}(0)A_j^{(1)}A_k^{(1)}A_l^{(1)} \\
&+\sum_{j=0}^{n}\sum_{k=0}^{j}\varphi_{jk}^{(h)}(0)(A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)})\Bigg]+A_g^{(3)},
\end{aligned}
$$

equations (34) become

$$
\begin{aligned}
x_3^{(i)} = &\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\varphi_{jkl}^{(i)}-\sum_{g=1}^{n}\sum_{h=1}^{n}\Delta_g^{(h)}\varphi_{jkl}^{(h)}(0)\xi_g^{(i)}\right] \\
&+\sum_{j=0}^{n}\sum_{k=0}^{j}\left[\varphi_{jk}^{(i)}-\sum_{g=1}^{n}\sum_{h=1}^{n}\Delta_g^{(h)}\varphi_{jk}^{(h)}(0)\xi_g^{(i)}\right][A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)}]+\sum_{j=1}^{n}A_i^{(3)}\xi_j^{(i)};
\end{aligned}
$$

which are simply

$$
\begin{aligned}
x_3^{(i)} = &\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}x_{jkl}^{(i)}A_j^{(1)}A_k^{(1)}A_l^{(1)} \\
(35)\qquad &+\sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(i)}(A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)})+\sum_{j=0}^{n}A_j^{(3)}x_j^{(i)},
\end{aligned}
$$

where the $A_j^{(3)}$ are the constants of integration, $A_0^{(3)} = 0$, and $x_{jkl}^{(i)}$ and $x_{jk}^{(i)}$ are the same functions of t which occur in (24) and (21). Since these functions vanish with t, the initial values of the $x_3^{(i)}$ are

$$
x_3^{(i)}(0) = \sum_{j=0}^{n}A_j^{(3)}x_j^{(i)}(0).
$$

The coefficients of the higher powers of μ can be determined in a similar manner. So far as they have been worked out, the solutions as power series in μ are

$$x_i = \left[\sum_{j=0}^{n} A_j^{(1)} x_j^{(i)}\right]\mu + \left[\sum_{j=0}^{n}\sum_{k=0}^{j} x_{jk}^{(i)} A_j^{(1)} A_k^{(1)} + \sum_{j=0}^{n} x_j^{(i)} A_j^{(2)}\right]\mu^2$$

$$(36) \qquad + \left[\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k} x_{jkl}^{(i)} A_j^{(1)} A_k^{(1)} A_l^{(1)} + \sum_{j=0}^{n}\sum_{k=0}^{j} x_{jk}^{(i)} (A_j^{(1)} A_k^{(2)} + A_j^{(2)} A_k^{(1)}) + \sum_{j=0}^{n} A_j^{(3)} x_j^{(i)}\right]\mu^3 + \cdots,$$

where $A_0^{(1)} = 1$ and $A_0^{(p)} = 0$, $(p = 2, \cdots, \infty)$ and all the other $A_j^{(p)}$ are undetermined. But equations (36) are precisely what one obtains on substituting in (17)

$$(37) \qquad \beta_0 = \mu, \qquad \beta_j = A_j^{(1)}\mu + A_j^{(2)}\mu^2 + A_j^{(3)}\mu^3 + \cdots \quad (j = 1, \cdots, n),$$

This, of course, should be expected, for if we take

$$0 < A_j^{(p)} < M,$$

where M is a positive constant, the substitution (37) converges for all values of $\mu < 1$, and consequently the substitution of (37) in (17) will give the solutions of the differential equations as power series in μ which converge for values of μ sufficiently small, and which reduce at $t = 0$ to

$$x_i(0) = \sum_{k=1}^{\infty}\sum_{j=0}^{n} A_j^{(k)} x_j^{(i)}(0)\,\mu^k.$$

But these are exactly the conditions under which (36) were developed. The two series are therefore identical.

Conditions for Periodic Solutions.

From the hypothesis as to the $\theta_{j\ldots}^{(i)}$ in the coefficients of the differential equations, i. e., that they are periodic functions of t with the period 2π, it follows that sufficient conditions that the solutions shall be periodic with the period $2k\pi$ (k an integer) are

$$x_i(2k\pi) = x_i(0) \qquad (i = 1, \cdots, n).$$

If the x_i all return to their original values, it is obvious from the differential equations that their first derivatives retake their initial values, and therefore all higher derivatives do likewise. Consequently under these conditions

$$x_i(t + 2k\pi) \equiv x_i(t) \qquad (i = 1, \cdots, n).$$

If we denote the difference between the value of a function at $t = 2k\pi$ and at $t = 0$ by a dash over the letter representing the function, e. g.,

$$\bar{x}_i = x_i\,(2k\pi) - x_i\,(0),$$

the conditions for periodicity as derived from (18) are

$$0 = \sum_{j=0}^{n} \bar{x}_j^{(i)} \beta_j + \sum_{j=0}^{n} \sum_{k=0}^{j} \bar{x}_{jk}^{(i)} \beta_j \beta_k + \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{l=0}^{k} \bar{x}_{jkl}^{(i)} \beta_j \beta_k \beta_l + \cdots. \tag{38}$$

As derived from (36), the conditions are

$$\begin{aligned} 0 \underset{\mu}{\equiv} & \left[\sum_{j=0}^{n} A_j^{(1)} \bar{x}_j^{(i)} \right] \mu + \left[\sum_{j=0}^{n} \sum_{k=0}^{j} \bar{x}_{jk}^{(i)} A_j^{(1)} A_k^{(1)} + \sum_{j=0}^{n} A_j^{(2)} \bar{x}_j^{(i)} \right] \mu^2 \\ & + \left[\sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{l=0}^{k} \bar{x}_{jkl}^{(i)} A_j^{(1)} A_k^{(1)} A_l^{(1)} + \sum_{j=0}^{n} \sum_{k=0}^{k} \bar{x}_{jk}^{(i)} (A_j^{(1)} A_k^{(2)} + A_j^{(2)} A_k^{(1)}) \right. \\ & \left. + \sum_{j=0}^{n} A_j^{(3)} \bar{x}_j^{(i)} \right] \mu^3 + \cdots. \end{aligned} \tag{39}$$

Since equations (39) can be derived from equations (38) by the substitution (37), it is clear that if the undetermined constants $A_j^{(p)}$ can be determined so as to satisfy (39), then the values of the β_j as defined in (37) will satisfy the conditions (38). That is, the determination of the constants $A_j^{(p)}$ so as to make the series (36) periodic is equivalent to a purely formal solution of the equations of condition (38). The convergency of the series thus obtained is thus reduced to a question in the theory of implicit functions: Under what conditions does a purely formal solution of a set of equations of the type (38) converge?

Let us suppose first that in the coefficient of μ^p every $A_j^{(p)}$ can be determined. This will be possible if the determinant $| \bar{x}_j^{(i)} | \neq 0$ $(i, j = 1, \cdots, n_j)$, and the determination is unique. But if this determinant is not zero equations (38) can also be solved uniquely, since $| \bar{x}_j^{(i)} |$ is its functional determinant.

The condition that the determinant

$$| \bar{x}_j^{(i)} | = | \bar{\xi}_j^{(i)} | \qquad (i, j = 1, \cdots, n)$$

shall not vanish is equivalent to the condition given by Poincaré,* but is somewhat simpler. The functional determinant, $| \partial\psi_i / \partial\beta_j |$ in his notation, is the functional determinant of the α_i obtained from (38) after transforming them by equations (15). It can be shown without trouble that the functional determinant of the α_i is the product of the two determinants $| \xi_j^{(i)}(0) |$ and $| \bar{\xi}_j^{(i)} |$, the first of which (see equation (4) et seq.) has the value unity. The second can be formed as soon as a fundamental set of solutions of (3) is known.

* *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 83.

Let us suppose that the roots of the fundamental equation for the solutions of (3) are all distinct. Then

$$\xi_j^{(i)} = e^{\lambda_j t}\psi_j^{(i)}(t),$$

where the $\psi_j^{(i)}(t)$ are periodic with the period 2π. The determinant

$$|\bar{\xi}_j^{(i)}| = |\psi_j^{(i)}(0)| \prod_{j=1}^{n}(e^{2k\lambda_j\pi} - 1) = \prod_{j=1}^{n}(e^{2k\lambda_j\pi} - 1) \quad (k \text{ an integer}),$$

can vanish if, and only if, $k\lambda_j \equiv 0 \bmod \sqrt{-1}$ for one or more values of j. The same result holds if two or more of the characteristic exponents λ_j are equal. Consequently periodic solutions with the period $2k\pi$ always exist if $k\lambda_j \not\equiv 0 \bmod \sqrt{-1}$ for all j. In particular, if $k = 1$ we have the theorem that, if none of the characteristic exponents λ_j are congruent to zero mod $\sqrt{-1}$, then there exists one and only one periodic solution with the period 2π.*

It is to be observed that the determinant $|\bar{\xi}_j^{(i)}|$ may vanish for $t = 2k\pi$ while it does not vanish for $t = 2\pi$. Indeed, if any of the λ_j are pure imaginaries and rational, there exist values of k for which the determinant vanishes.

If the determinant $|\bar{\xi}_j^{(i)}|$ vanishes, then the constants $A_j^{(p)}$ cannot all be determined by the periodicity conditions on the coefficients of μ^p, and in general the solutions will no longer be unique. Nevertheless, it may still be possible to determine the $A_j^{(p)}$ so as to satisfy the periodicity conditions, and the solutions thus obtained will converge provided the determination becomes unique at any step; for it is shown in a paper entitled *A method for determining the solutions of a set of analytic functions in the neighborhood of a branch point* † that a formal solution of a system of equations of the type of (38) as a power series in integral or fractional powers of μ converges provided the solution is not multiple identically in μ. If the solution is such a multiple solution, then at no step will the determination of the constants $A_j^{(p)}$ become unique, and the question of the convergence of such coincident branches remains open. We have then the

THEOREM: *If*

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n; \mu; t) \qquad (i = 1, \cdots, n)$$

is a system of differential equations in which the f_i are expansible as power series in $x_1, \cdots, x_n$, and μ, vanishing for $x_1 = \cdots = x_n = \mu = 0$, with coefficients which are uniform, continuous and periodic functions of t with the period 2π; and if the f_i converge for $0 \leqq t \leqq 2\pi$ when $x_j < \rho_i$, $\mu < r$, then the solutions $x_i(t)$ are

* This theorem was given by POINCARÉ, loc. cit., p. 181.

† MACMILLAN, Mathematische Annalen. The paper is in type and will appear in 1912.

expansible as power series in μ, or any fractional power of μ, which converge for all t in the interval $0 \leqq t \leqq 2k\pi$ provided $|\mu|$ is sufficiently small. If the constants of integration arising at each step can be determined so as to make the solution formally periodic with the period $2k\pi$, then the solution so determined will be periodic and converge for all finite values of t provided $|\mu|$ is sufficiently small.

If periodic solutions exist as power series in fractional powers of μ, but not in integral powers, then it will not be possible to determine the constants $A_j^{(p)}$ in (36) so as to satisfy the periodicity conditions, but on taking

$$A_j^{(1)}\mu = \beta_j \qquad (j = 1, \cdots, n),$$

$$A_j^{(p)} = 0 \qquad (p = 2, \cdots \infty),$$

we pass at once to the general equations of existence (38) in $\beta_1, \cdots, \beta_n$, which are more convenient and much simpler than the corresponding equations expressed in terms of the initial values of the x_i. The forms of the periodic solutions, if they exist at all as power series in integral or fractional powers of μ, are then readily found. But if the construction of the solutions can be made in either integral or fractional powers of μ, one is assured of the convergence of the solutions so obtained for all values of t provided the modulus of μ is sufficiently small.

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